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DESIGN OF PINNED BLADES FOR VIBRATION

B. F. Shorr

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ABSTRACT

Development of a general method for calculating the vibrations of twisted pinned blades under combined bending and torsion. The analysis extends previous results obtained for rigid blades.

In reference 1 consideration was given to the kinematics of pinned /292* blades in the presence of structural clearance in the pinned joint with 2 degrees of freedom and approximate estimates were obtained for the first form of vibrations. In this article a general method for the calculation of combined bending and torsional vibrations of twisted pinned blades is presented. It is an extension of an analogous method developed earlier for the rigidly mounted blades (ref. 2).

The principal design of a pinned blade is shown in figure 1. Subsequently it is assumed that the hinge axis is directed parallel to the axis of the rotor. The mass of the pin is negligibly small in comparison to the mass of the /293
blade. Vibrations occur at low amplitude, which enables us to use linearized equations and to limit our considerations of the motion of the blade during

*Numbers given in margin indicate pagination in original foreign text.

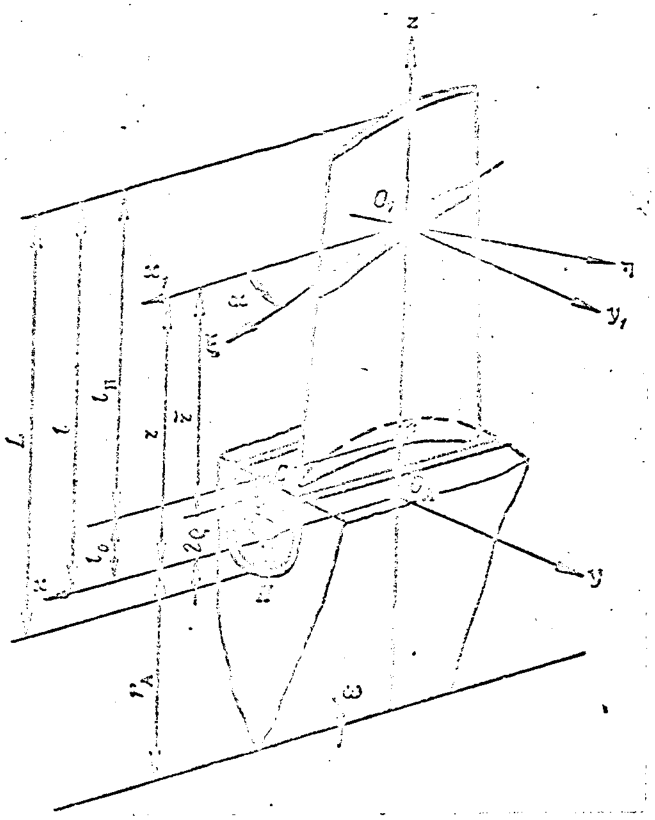

Figure 1. Schematic diagram of pinned blade.

rolling in the pinned joint without slippage (ref. 1). Under these conditions the motion of the blade as a solid in the plane of the rotation of the rotor is

$$v_T = z\varphi_1 + \tilde{z}\varphi_2, \tag{1}$$

where φ_1 and φ_2 are angles of rotation of the blade with respect to the pin of the hinge and of the pin with respect to the disk, $\tilde{z} = z - 2\rho$, and ρ is the pin radius.

The coordinates of the contact points of the blade with pin B and the pin disk D in the deflected position are equal to (ref. 1)

$$\begin{aligned} &y_B = -\rho(\gamma_1\varphi_1 + 2\varphi_2), \quad z_B = 0, \quad \tilde{z}_B = -2\rho, \quad r_B = r_A, \\ &y_D = -\rho\gamma_2\varphi_2, \quad z_D = 2\rho, \quad \tilde{z}_D = 0, \quad r_D = r_C, \end{aligned} \tag{2}$$

where (fig. 2)

$$\gamma_1 = \frac{\varrho_1}{\varrho_1 - \varrho}, \quad \gamma_2 = \frac{\varrho_2}{\varrho_2 - \varrho} \tag{3}$$

are dimensionless parameters equal to the reciprocals of the respective clearances in the pinned joint; in the general case $1 \leq \gamma_{1,2} \leq \infty$. /294

In the investigation of the lower forms of bending and torsional vibrations of twisted blades it is generally possible to neglect the effect of longitudinal inertia forces and flexure in the plane of greatest rigidity. Here the relationships between the components of the elastic curvature in the lowest rigidity

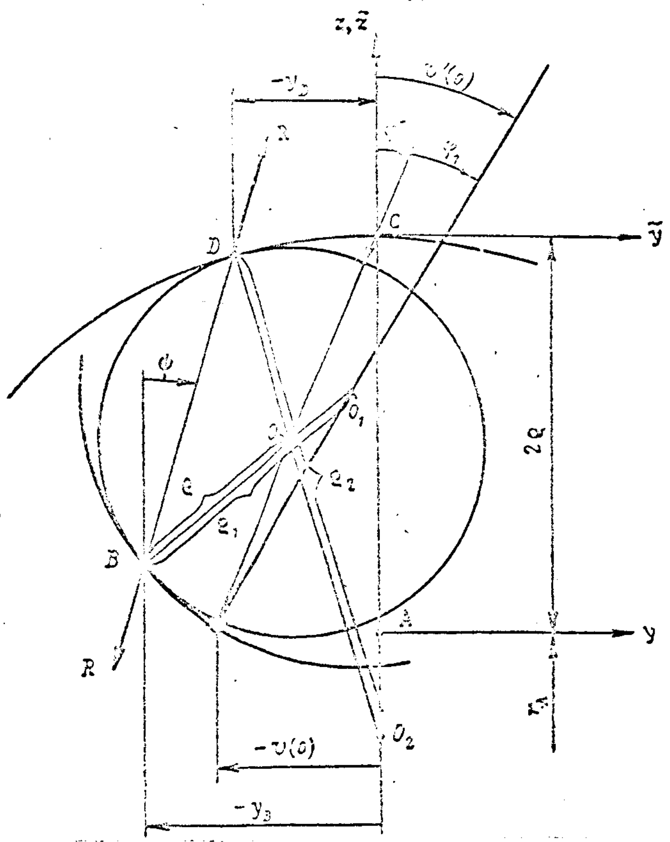

Figure 2. Diagram of pinned blade with two degrees of freedom in a tilted position.

plane χ_ξ and torsion τ along with bending moment in the same plane M_ξ and torsional moment M_z in accordance with the theory of twisted rods (refs. 2,3) are of the following form

$$\begin{aligned} -\chi_\xi &= \delta_{11} M_\xi + \delta_{12} M_z, \\ \tau &= \delta_{12} M_\xi + \delta_{22} M_z, \end{aligned} \tag{4}$$

where δ_{11}, δ_{12} and δ_{22} are yielding coefficients which are defined by the formulae, cited in reference 2, where the following notation was adopted: $a_\eta = \delta_{11}$, $a_{\eta\tau} = -\delta_{12}$, $a_\tau = \delta_{22}$ and $\varkappa_\eta = \chi_\xi$.

Since the components of bending deformation χ_ξ, χ_η and torsion τ are related to displacement components u, v and rotation θ in the following manner

$$\begin{aligned} \chi_\xi &= -u'' \sin\alpha + v'' \cos\alpha, \\ \chi_\eta &= u'' \cos\alpha + v'' \sin\alpha, \\ \tau &= \theta', \end{aligned} \tag{5}$$

where the superscript designates derivatives along the z coordinate, then when $\chi_\eta \approx 0$ and at boundary conditions in the pinned joint (see fig. 2) we have

$$\begin{aligned} u(0) &= u'(0) = \theta(0) = 0, \\ v(0) &= v_2(0) = -2\zeta\varphi_2, \\ v'(0) &= v_2'(0) = \varphi_1 + \varphi_2 \end{aligned} \tag{6}$$

and we can obtain the inverse relationships

$$\begin{aligned} u &= -\int_0^z \int_0^z \chi_\xi \sin\alpha \, dz^2, \\ v &= z\varphi_1 + \tilde{z}\varphi_2 + \int_0^z \int_0^z \chi_\xi \cos\alpha \, dz^2, \\ \theta &= \int_0^z \tau \, dz. \end{aligned} \tag{7}$$

In the case of harmonic vibrations of the blade in the centrifugal force field (without taking into account the effect of the inertia of rotation with respect to the x and y axes and coriolis forces) the amplitudes of variable loads will be

$$
\begin{aligned}
q_x &= \rho_{л} p^2 F u, \\
q_y &= \rho_{л} (p^2 + \omega^2) F v, \\
m_z &= \rho_{л} [p^2 J_p + \omega^2 (J_y - J_x)] \theta, \\
m_x &= m_y = 0,
\end{aligned}
\qquad (8)
$$

л = blade

where $\rho_{л}$ is the density of the material of the blade;

p is cyclic frequency of vibrations,

ω is angular velocity of rotation of the blade;

F is area;

$J_p = J_x + J_y$ is the polar moment of the inertia of the cross section.

The equation of interest is the equilibrium of the element of the blade, linear with respect to the amplitudes of the variable rod components, and displacements have the following form /295

$$
\begin{aligned}
m'_x + q_y + M''_x + (P_z v')' &= 0, \\
m'_y - q_x + M''_y - (P_z u')' &= 0 \\
m_z + M'_z &= 0,
\end{aligned}
\qquad (9)
$$

where

$$
\begin{aligned}
M_\xi &= M_x \cos\alpha + M_y \sin\alpha, \\
M_\eta &= -M_x \sin\alpha + M_y \cos\alpha
\end{aligned}
\qquad (10)
$$

and P_z is centrifugal force of the part of the blade from the cross section z to the free end, equal to $P_z = \rho_{л} \omega^2 P_1$, where

$$P_1 = \int_z^l rF\,dz. \tag{11}$$

By introducing (8) into the equilibrium equations (9) and integrating, taking into account the following boundary conditions at the free end of the blade

$$M_x(l) = M'_x(l) = M_y(l) = M'_y(l) = M_z(l) = 0, \tag{12}$$

we obtain

$$\begin{aligned} M_x &= -\varrho_{\text{л}} p^2 \int_z^l \int_z^l Fv\,dz^2 + \varrho_{\text{л}} \omega^2 r \int_z^l P_1 \left(\frac{v}{r}\right)' dz, \\ M_y &= \varrho_{\text{л}} p^2 \int_z^l \int_z^l Fu\,dz^2 - \varrho_{\text{л}} \omega^2 \int_z^l P_1 u'\,dz, \\ M_z &= \varrho_{\text{л}} p^2 \int_z^l J_p \theta\,dz + \varrho_{\text{л}} \omega^2 \int_z^l (J_y - J_x)\theta\,dz. \end{aligned} \tag{13}$$

By substituting expression (7) into (13) and converting to moments M_ξ and M_z we obtain the following system of integral equations which describe the bending and torsional vibrations of a twisted pin mounted blade with two degrees of freedom in the pinned joint:

$$\begin{gathered} M_\xi = -\varphi_1 (p^2 B_{p1} - \omega^2 B_{\omega_1}) - \varphi_2 (p^2 B_{p2} - \omega^2 B_{\omega_2}) + p^2 \Phi_\xi - \omega^2 \Psi_\xi, \\ M_z = p^2 \Phi_z - \omega^2 \Psi_z, \end{gathered} \tag{14}$$

where

$$B_{p1} = \varrho_{\text{л}} \cos\alpha \int_z^l \int_z^l Fz\,dz^2, \quad B_{p2} = \varrho_{\text{л}} \cos\alpha \int_z^l \int_z^l F\tilde{z}\,dz^2,$$

$$B_{\omega_1} = \varrho_{\text{л}} r r_A \cos\alpha \int_z^l \frac{P_1}{r^2}\,dz, \quad B_{\omega_2} = \varrho_{\text{л}} r r_C \cos\alpha \int_z^l \frac{P_1}{r^2}\,dz,$$

$$\Phi_\xi = -\varrho_л \left[\cos\alpha \int_z^l \int_z^l F \int_0^z \int_0^z \chi_\xi \cos\alpha \, dz^4 + \sin\alpha \int_z^l \int_z^l F \int_0^z \int_0^z \chi_\xi \sin\alpha \, dz^4 \right],$$

$$\Psi_\xi = -\varrho_л \left[r\cos\alpha \int_z^l \frac{P_1}{r^2} \int_0^z r\chi_\xi \cos\alpha \, dz^2 + \sin\alpha \int_z^l P_1 \int_0^z \chi_\xi \sin\alpha \, dz^2 \right], \qquad (15)$$

$$\Phi_z = \varrho_л \int_z^l J_p \int_0^z \tau \, dz^2,$$

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$$\Psi_z = -\varrho_л \int_z^l (J_y - J_x) \int_0^z \tau \, dz^2,$$

here χ_ξ and τ are expressed through M_ξ and M_z by equations (4).

For the specific case of a slightly twisted blade ($\delta_{12} = 0$, $\delta_{11} = (EJ_\xi)^{-1}$, $\delta_{22} = (GT)^{-1}$) with a stationary point hinge ($\varrho = 0$, $\varphi_2 = 0$) equations (14) coincide with the equation obtained earlier by I. A. Birger (ref. 4).

Angles φ_1 and φ_2 are determined on the basis of the condition that the moments of all forces acting on the blade with respect to points of contact in the pinned joint are equal to zero. The variable loading moment (8) and constant load $q_z^0 = \varrho_л \omega^2 Fr$, with respect to a random axis passing through point y^*, z^* normal to the plane yz, are equal to

$$M^* = \int_L [q_z^0 (v - y^*) - q_y (z - z^*)] \, dz, \qquad (16)$$

where the integration is conducted under the whole length of the blade L. The mass of the pin is neglected.

By introducing expression (8) for q_y and expression (7) for v into equation (16) we obtain

$$
\begin{aligned}
M^* = & -\varphi_1 \varrho_л [p^2 (J_A - z^* S_A) - \omega^2 r^* S_A] - \\
& - \varphi_2 \varrho_л [p^2 (J_C - \tilde{z}^* S_C) - \omega^2 r^* S_C] - \\
& - \varrho_л p^2 \int\limits_L F (r - r^*) \int\limits_{z^*}^{z} \int\limits_0^{z} \chi_\xi \cos \alpha \, dz^3 + \\
& + \varrho_л \omega^2 r^* \int\limits_L F \int\limits_0^{z} \int\limits_0^{z} \chi_\xi \cos \alpha \, dz^3 - y^* \varrho_л \omega^2 P_{1л},
\end{aligned} \tag{17}
$$

where J_A, S_A and J_C, S_C are moments of inertia and static moments for the whole volume of the blade with respect to contact points A and C respectively. These moments are determined by the formulas

$$
\begin{aligned}
& J_A = \int\limits_L (F z^2 + J_x) \, dz, \quad J_C = \int\limits_L (F \tilde{z}^2 + J_x) \, dz, \\
& S_A = \int\limits_L F z \, dz, \quad S_C = \int\limits_L F \tilde{z} \, dz.
\end{aligned} \tag{18}
$$

The quantity $P_{1л}$ is equal to $P_{1л} = \int\limits_L rF \, dz$.

Designating

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$$
\begin{aligned}
& \Phi_L = -\varrho_л \int\limits_L \int\limits_z^{l} F \int\limits_0^{z} \int\limits_0^{z} \chi_\xi \cos \alpha \, dz^4 \\
& (\Phi_L = \Phi_\xi \text{ when } \alpha = 0, \ z = z_л) \\
& \Psi_L = -\varrho_л r_н \int\limits_L F \int\limits_0^{z} \int\limits_0^{z} \chi_\xi \cos \alpha \, dz^3 = -\varrho_л r_н \int\limits_L \frac{P_1}{r^2} \int\limits_0^{z} r \chi_\xi \cos \alpha \, dz^2, \\
& (\Psi_L = \Psi_\xi \text{ when } \alpha = 0, \ z = z_н),
\end{aligned} \tag{19}
$$

where the subscript "H" refers to the low point on the blade ($L = l - z_H$, $z_H < 0$), and taking into account the fact that

$$
\varrho_л \int\limits_L F (r - r^*) \int\limits_0^{z} \int\limits_0^{z} \chi_\xi \cos \alpha \, dz^3 = -\left[\Phi_L + \left(1 - \frac{r^*}{r_н}\right) \Psi_L \right],
$$

we may write equation (17) in the form

$$\begin{gathered} M^* = -\varphi_1 \varrho_\text{л} [p^2 (J_A - z^* S_A) - \omega^2 r^* S_A] - \\ - \varphi_2 \varrho_\text{л} [p^2 (J_C - \tilde{z}^* S_C) - \omega^2 r^* S_C] + p^2 \left[\Phi_L + \left(1 - \frac{r^*}{r_\text{н}}\right) \Psi_L \right] - \\ - \omega^2 \frac{r^*}{r_\text{н}} \Psi_L - y^* \varrho_\text{л} \omega^2 P_{1\text{л}}. \end{gathered} \tag{20}$$

By introducing expressions for coordinates of the points of contact (2) into equation (20) we find

$$\begin{gathered} M_B = -\varphi_2 \varrho_\text{л} [p^2 J_A - \omega^2 (r_A S_A + \varrho \gamma_1 P_{1\text{л}})] - \\ - \varphi_2 \varrho_\text{л} [p^2 (J_C + 2\varrho S_C) - \omega^2 (r_A S_C + 2\varrho P_{1\text{л}})] + \\ + p^2 \left[\Phi_L + \left(1 - \frac{r_A}{r_\text{н}}\right) \Psi_L \right] - \omega^2 \frac{r_A}{r_\text{н}} \Psi_L = 0, \\ M_D = -\varphi_1 \varrho_\text{л} [p^2 (J_A - 2\varrho S_A) - \omega^2 r_C S_A] - \\ - \varphi_2 \varrho_\text{л} [p^2 J_C - \omega^2 (r_C S_C + \varrho \gamma_2 P_{1\text{л}})] + \\ + p^2 \left[\Phi_L + \left(1 - \frac{r_C}{r_\text{н}}\right) \Psi_L \right] - \omega^2 \frac{r_C}{r_\text{н}} \Psi_L = 0. \end{gathered} \tag{21}$$

Let us denote the following explicit relationships

$$\begin{gathered} S_A - S_C = 2\varrho V, \\ J_A - J_C = 2\varrho (S_A + S_C), \\ r_C S_A = r_A S_C + 2\varrho P_{1\text{л}}, \end{gathered} \tag{22}$$

where V, the volume of the whole blade, is equal to $V = \int_L F\,dz$.

Generally, the mass of the blade located below point A (i.e., when $r < r_A$) is negligibly small in comparison with the rest of the mass of the blade and $\varrho/l \ll 1$. In this case $l \approx L$ and

$$
\begin{aligned}
&B_{p1A} \approx \varrho_{\pi} J_A, && B_{p2C} \approx \varrho_{\pi} J_C, \\
&B_{\omega 1A} \approx \varrho_{\pi} r_A S_A, && B_{\omega 2C} \approx \varrho_{\pi} r_C S_C, \\
&B_{p1C} \approx \varrho_{\pi}(J_A - 2\varrho S_A), && B_{p2A} \approx \varrho_{\pi}(J_C + 2\varrho S_C), \\
& && (B_{p1C} = B_{p2A}), \\
&B_{\omega 1C} \approx \varrho_{\pi} r_A S_C, && B_{\omega 2A} \approx \varrho_{\pi} r_C S_A.
\end{aligned}
\tag{23}
$$

In this case the inner bending moment along the $r = r_A$ and $r = r_C$ /298 crosssections according to (14) will be equal to

$$
\begin{gathered}
M_A = -\varphi_1 \varrho_{\pi}(p^2 J_A - \omega^2 r_A S_A) - \\
-\varphi_2 \varrho_{\pi}[p^2(J_C + 2\varrho S_C) - \omega^2(r_A S_C + 2\varrho P_{1\pi})] + p^2 \Phi_A - \omega^2 \Psi_A, \\
M_C = -\varphi_1 \varrho_{\pi}[p^2(J_A - 2\varrho S_A) - \omega^2(r_C S_A - 2\varrho P_{1\pi})] - \\
-\varphi_2 \varrho_{\pi}(p^2 J_C - \omega^2 r_C S_C) + p^2 \Phi_C - \omega^2 \Psi_C,
\end{gathered}
\tag{24}
$$

where $\Phi_A = \Phi_\xi$, $\Psi_A = \Psi_\xi$ where a = 0, z = 0 and $\Phi_C = \Phi_\xi$, $\Psi_C = \Psi_\xi$ when a = 0, $\tilde{z} = 0$.

If the bending rigidity of the tail end is significantly higher than the rigidity of the fin, then, assuming that when $z < l_0$, quantity $\chi_\xi = 0$, we find that operators $\Phi_x(r)$ and $\Psi_x(r)$ in the random crosssection of the tail end are expressed through their values in the base cross section of the fin ($\Phi_{x0} = \Phi_\xi$, $\Psi_{x0} = \Psi$ when a = 0, $z = l_0$) by the following formulas

$$
\begin{gathered}
\Phi_x(r) = \Phi_{x0} + \left(1 - \frac{r}{r_0}\right) \Psi_{x0}, \\
\Psi_x(r) = \frac{r}{r_0} \Psi_{x0},
\end{gathered}
\tag{25}
$$

from which it follows that

$$\Phi_A = \Phi_L + \left(1 - \frac{r_A}{r_H}\right)\Psi_L, \qquad (26)$$
$$\Psi_A = \frac{r_A}{r_H}\Psi_L$$

and analogous expressions can be written for Φ_C and Ψ_C.

Comparing (24) with (21) and taking (26) into account, we find that conditions $M_B = M_D = 0$ may be represented in the form

$$M_A = -\omega^2\varphi_1 B_{\rho 1} = -P_{z\,\text{л}} h_A, \qquad (27)$$
$$M_C = -\omega^2(\varphi_1 B_{\rho 0} + \varphi_2 B_{\rho 2}) = -P_{z\,\text{л}} h_C,$$

where

$$B_{\rho 0} = 2\rho\rho_{\text{л}} P_{1\text{л}}, \qquad h_A = \rho\gamma_1\varphi_1, \qquad (28)$$
$$B_{\rho 1} = \gamma_1\rho\rho_{\text{л}} P_{1\text{л}}, \qquad h_C = \rho(2\varphi_1 + \gamma_2\varphi_2),$$
$$B_{\rho 2} = \gamma_2\rho\rho_{\text{л}} P_{1\text{л}}, \qquad P_{z\,\text{л}} = \rho_{\text{л}}\omega^2 P_{1\text{л}}.$$

It can be seen from Figure 3 that $h_A = v_A - y_B$ and $h_C = v_C - y_D$ are equal to the distances in the bent position between points of contact and points on the axis of the blade located on the same radii.

By subtracting in equation (21) quantity M_B from M_D or in equation (27) quantity M_A from M_C, when $\rho \neq 0$ we obtain respectively

$$\varphi_1\rho_{\text{л}}[p^2 S_A + \omega^2(S_A - 0{,}5\gamma_1 P_{1\text{л}})] + \varphi_2\rho_{\text{л}}\{p^2 S_C + \qquad (29)$$
$$+ \omega^2[S_C - (1 - 0{,}5\gamma_2)P_{1\text{л}}]\} = \frac{\Psi_L}{r_H}(p^2 + \omega^2)$$

or (fig. 4)

$$P_{y\,\text{л}} \approx Q_A = \frac{M_C - M_A}{2\rho} + P_{z\,\text{л}}\frac{v_C - v_A}{2\rho} = \qquad (30)$$
$$= P_{z\,\text{л}}[\varphi_2 + 0{,}5(\gamma_1\varphi_1 - \gamma_2\varphi_2)].$$

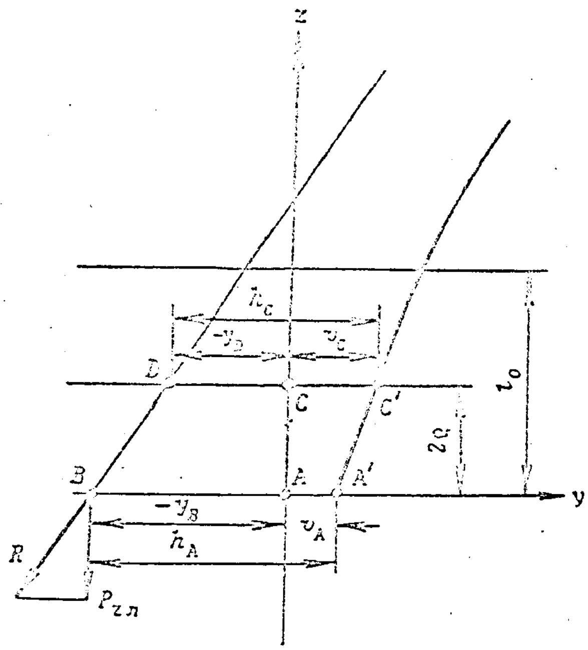

Figure 3. Formulation of boundary conditions in the pin joint.

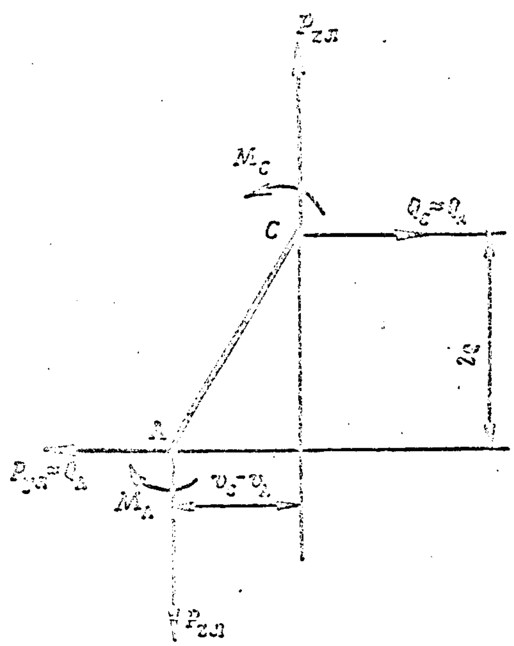

Figure 4. Determination of the transverse force Q_A.

It can be shown (ref. 1) that quantity

$$\psi = \varphi_2 + 0{,}5\,(\gamma_1\varphi_1 - \gamma_2\varphi_2) \tag{31}$$

determines the slope of line BD connecting points of contact in the /300
deflected position, and consequently equations (29) and (30) reflect the condition

$$\varphi_R = \frac{P_{yA}}{P_{zA}} = \psi, \tag{32}$$

i.e., the force of reaction R passes through the points of contact (see fig. 2).

For calculation in the general case one can make use of equation (29) and one of equations (21). In the specific case when the pin is stationary with respect to the blade ($\varphi_1 = 0$, $\gamma_1 = \infty$) only the second equation (21) is meaningful, from which it follows that

$$\varphi = \varphi_2 = \frac{p^2\Phi_C - \omega^2\Psi_C}{p^2 B_{p2C} - \omega^2(B_{\omega 2C} + B_{\varphi 2})}. \tag{33}$$

When the pin is immobile with respect to the disk ($\gamma_2 = \infty$, $\varphi_2 = 0$), from the first equation (21) it follows that

$$\varphi = \varphi_1 = \frac{p^2\Phi_A - \omega^2\Psi_A}{p^2 B_{p1A} - \omega^2(B_{\omega 1A} + B_{\varphi 1})}. \tag{34}$$

For the rigid blade the operators $\Phi \equiv \Psi \equiv 0$ and the system (21) may be reduced by substitution of variables to the system of equations obtained in reference 1 for the pendulum form of oscillations (without taking into account the mass of the fin).

Let us now consider the analysis of orthogonality conditions for the m^{th} and n^{th} forms of vibrations. The orthogonality conditions for the displacements within the framework of the considered theory for twisted rods has, according to (ref. 3), the following form

$$\int_l [F(u_m u_n + v_m v_n) + J_p \theta_m \theta_n]\, dz = 0 \quad (p_n \neq p_m, \quad \omega_n = \omega_m). \tag{35}$$

Integrating (35) by parts and taking into account the boundary conditions of (6) and (12), in the case of a rigid tail section we obtain ($\chi_\xi = 0$ when $z < l_0$)

$$\int_{l_0}^{l} (M_{pym} u''_n - M_{pxm} v''_n + M_{pzm} \theta')\, dz - M_{pAm} \varphi_{1n} - M_{pCm} \varphi_{2n} = 0, \tag{36}$$

where

$$\begin{aligned} M_{py} &= \varrho_a p^2 \int_z^l \int_z^l F u\, dz^2, \\ M_{px} &= -\varrho_a p^2 \int_z^l \int_z^l F v\, dz^2, \\ M_{pz} &= \varrho_a p^2 \int_z^l J_p \theta\, dz \end{aligned} \tag{37}$$

are the internal moments due to inertia loads caused by complete displacements of the blade, and

$$\begin{aligned} M_{pA} &= -\varrho_a p^2 \int_l F v z\, dz, \\ M_{pC} &= -\varrho_a p^2 \int_l F v \tilde{z}\, dz \end{aligned} \tag{38}$$

are the moments of inertia loads with respect to the axes passing through points A and C parallel to the x axis respectively.

When $\chi_\eta \approx 0$, it follows from (36) that /301

$$\int_{l_0}^{l} (-M_{p\xi m}\gamma_{\xi n} + M_{pzm}\tau_n)\, dz - M_{pAm}\varphi_{1n} - M_{pCm}\varphi_{2n} = 0. \tag{39}$$

On the basis of expression (14) one may write

$$\begin{aligned} M_{p\xi} &= p^2\widetilde{\Phi}_\xi, \quad M_{pA} = p^2\widetilde{\Phi}_A, \\ M_{pz} &= p^2\Phi_z, \quad M_{pC} = p^2\widetilde{\Phi}_C, \end{aligned} \tag{40}$$

where

$$\begin{aligned} \widetilde{\Phi}_\xi &= \Phi_\xi - (\varphi_1 B_{p1} + \varphi_2 B_{p2}), \\ \widetilde{\Phi}_A &= \Phi_A - \varrho_a[\varphi_1 J_A + \varphi_2(J_C + 2\zeta S_C)] \approx \Phi_A - (\varphi_1 B_{p1A} + \varphi_2 B_{p2A}), \\ \widetilde{\Phi}_C &= \Phi_C - \varrho_a[\varphi_1(J_A - 2\zeta S_A) + \varphi_2 J_C] \approx \Phi_C - (\varphi_1 B_{p1C} + \varphi_2 B_{p2C}). \end{aligned} \tag{41}$$

Thus, the orthogonality condition becomes

$$\int_{l_0}^{l} (-\widetilde{\Phi}_{\xi m}\gamma_{\xi n} + \Phi_{zm}\tau_n)\, dz - \widetilde{\Phi}_{Am}\varphi_{1n} - \widetilde{\Phi}_{Cm}\varphi_{2n} = 0. \tag{42}$$

The system of integral equations (14) with additional boundary conditions (21) and orthogonality conditions (42) may be solved by the method of succesive approximations. The order of calculation of the i^{th} approximation of the n^{th} form of vibrations is as follows. From the earlier found or preassigned values of moments of the $(i-1)^{th}$ approximation $M_{\xi n}^{i-1}$, M_{zn}^{i-1}, which satisfy the orthogonality conditions for all of the previous forms through the use of formula (15) we can find the values of the integral operators $\Phi_{\xi n}^{i-1}$, $\Psi_{\xi n}^{i-1}$, Φ_{zn}^{i-1}, Ψ_{zn}^{i-1}. In order for the moments for the i^{th} approximation and the corresponding values of angles to satisfy orthogonality conditions we must determine them in the form

$$M^i_{\xi n} = -\bar{\varphi}^i_{1n}(p^2_{ni}B_{p1} - \omega^2 B_{\omega 1}) - \bar{\varphi}^i_{2n}(p^2_{ni}B_{p2} - \omega^2 B_{\omega 2}) + p^2_{ni}\Phi^{i-1}_{\xi n} - \omega^2 \Psi^{i-1}_{\xi n} + \sum_{m=1}^{n-1} \beta^i_m M_{\xi m}, \tag{43}$$

and

$$M^i_{zn} = p^2_{ni}\Phi^{i-1}_{zn} - \omega^2 \Psi^{i-1}_{zn} + \sum_{m=1}^{n-1} \beta^i_m M_{zm},$$

$$\varphi^i_{1n} = \bar{\varphi}^i_{1n} + \sum_{m=1}^{n-1} \beta^i_m \varphi_{1m}, \tag{44}$$

$$\varphi^i_{2n} = \bar{\varphi}^i_{2n} + \sum_{m=1}^{n-1} \beta^i_m \varphi_{2m},$$

where $M_{\xi m}$, M_{zm} are moments and φ_{1m}, φ_{2m} are the angles of the m^{th} form of vibrations which are considered to be known; β^i_m are coefficients which are found from orthogonality conditions; the values $p^2_{ni} = (p^i_n)^2$ and $\bar{\varphi}^i_{1\eta}$, $\bar{\varphi}^i_{2\eta}$ are still random.

According to (ref. 4) the following deformation components correspond to moments in equation (43) /302

$$\chi^i_{\xi n} = \delta_{11}\bar{\varphi}^i_{1n}(p^2_{ni}B_{p1} - \omega^2 B_{\omega 1}) + \delta_{11}\bar{\varphi}^i_{2n}(p^2_{ni}B_{p2} - \omega^2 B_{\omega 2}) + p^2_{ni}K^{i-1}_{\chi n} - \omega^2 L^{i-1}_{\chi n} + \sum_{m=1}^{n-1} \beta^i_m \chi_{\xi m}, \tag{45}$$

$$\tau^i_n = -\delta_{12}\bar{\varphi}^i_{1n}(p^2_{ni}B_{p1} - \omega^2 B_{\omega 1}) - \delta_{12}\bar{\varphi}^i_{2n}(p^2_{ni}B_{p2} - \omega^2 B_{\omega 2}) + p^2_{ni}K^{i-1}_{\tau n} - \omega^2 L^{i-1}_{\tau n} + \sum_{m=1}^{n-1} \beta^i_m \tau_m,$$

where

$$-K_\chi = \delta_{11}\Phi_\xi + \delta_{12}\Phi_z, \qquad -L_\chi = \delta_{11}\Psi_\xi + \delta_{12}\Psi_z,$$
$$K_\tau = \delta_{12}\Phi_\xi + \delta_{22}\Phi_z, \qquad L_\tau = \delta_{12}\Psi_\xi + \delta_{22}\Psi_z, \tag{46}$$

and $\chi_{\xi m}$, τ_m are known components of deformation of the preceding m^{th} form. By substituting equation (45) into (n-1) orthogonality conditions

$$\int_{l_0}^{l}(-\widetilde{\Phi}_{\xi m}\chi_{\xi n}^{i}+\Phi_{zm}\tau_n^{i})\,dz-\widetilde{\Phi}_{Am}\left(\overline{\varphi}_{1n}^{i}+\sum_{m=1}^{n-1}\beta_m^{i}\varphi_{1m}\right)-$$
$$-\widetilde{\Phi}_{Cm}\left(\overline{\varphi}_{2n}^{i}+\sum_{m=1}^{n-1}\beta_m^{i}\varphi_{2m}\right)=0, \tag{47}$$

where $\widetilde{\Phi}_{\xi m}$, Φ_{zm}, $\widetilde{\Phi}_{Am}$, $\widetilde{\Phi}_{Cm}$, φ_{1m}, φ_{2m} are known quantities, we obtain

$$\int_{l_0}^{l}\Big\{\widetilde{K}_{\chi m}\,[\overline{\varphi}_{1n}^{i}\,(p_{ni}^2 B_{p1}-\omega^2 B_{\omega 1})+\overline{\varphi}_{2n}^{i}\,(p_{ni}^2 B_{p2}-\omega^2 B_{\omega 2})]+$$
$$+p_{ni}^2(-\widetilde{K}_{\chi m}\Phi_{\xi n}^{i-1}+\widetilde{K}_{\tau m}\Phi_{zn}^{i-1})-\omega^2(-\widetilde{K}_{\chi m}\Psi_{\xi n}^{i-1}+\widetilde{K}_{\tau m}\Psi_{zn}^{i-1})+ \tag{48}$$
$$+\beta_m^{i}(-\chi_{\xi m}\widetilde{\Phi}_{\xi m}+\tau_m\Phi_{zm})\Big\}\,dz-\widetilde{\Phi}_{Am}(\overline{\varphi}_{1n}^{i}+\beta_m^{i}\varphi_{1m})-$$
$$-\widetilde{\Phi}_{Cm}(\overline{\varphi}_{2n}^{i}+\beta_m^{i}\varphi_{2m})=0.$$

where the following representation is observed

$$\left.\begin{aligned}-\widetilde{K}_{\chi}&=\delta_{11}\widetilde{\Phi}_{\xi}+\delta_{12}\Phi_{z},\\ \widetilde{K}_{\tau}&=\delta_{12}\widetilde{\Phi}_{\xi}+\delta_{22}\Phi_{z}\end{aligned}\right\} \tag{49}$$

and in view of orthogonality conditions of all (n-1) forms among themselves the following is taken into account

$$\int_{l_0}^{l}(-\widetilde{\Phi}_{\xi m}\chi_{\xi k}+\Phi_{zm}\tau_k)\,dz-\widetilde{\Phi}_{Am}\varphi_{1k}-\widetilde{\Phi}_{Cm}\varphi_{2k}=0$$
$$\begin{pmatrix}k,\ m=1,2,\ldots,n-1\\ k\neq m\end{pmatrix}, \tag{50}$$

as well as the fact that according to (49)

$$\begin{aligned}-\widetilde{\Phi}_{\xi m}K_{\chi n}+\Phi_{zm}K_{\tau n}&\equiv-\widetilde{K}_{\chi m}\Phi_{\xi n}+\widetilde{K}_{\tau m}\Phi_{zn},\\ -\widetilde{\Phi}_{\xi m}L_{\chi n}+\Phi_{zm}L_{\tau n}&\equiv-\widetilde{K}_{\chi m}\Psi_{\xi n}+\widetilde{K}_{\tau m}\Psi_{zn}.\end{aligned} \tag{51}$$

From (48) we find the following expression for the coefficient β^i_m: /303

$$\beta^i_m = p^2_{ni}\beta^i_{m\Phi} - \omega^2\beta^i_{m\Psi} - \bar{\varphi}^i_{1n}(p^2_{ni}\beta_{mp1} - \omega^2\beta_{m\omega1} - \beta_{m\varphi1}) - \\ - \bar{\varphi}^i_{2n}(p^2_{ni}\beta_{mp2} - \omega^2\beta_{m\omega2} - \beta_{m\varphi2}), \tag{52}$$

where

$$\begin{gathered} \beta^i_{m\Phi} = c_m \int_{l_0}^{l} (\widetilde{K}_{\chi m}\Phi^{i-1}_{\xi n} - \widetilde{K}_{\tau m}\Phi^{i-1}_{zn})\,dz, \\ \beta^i_{m\Psi} = c_m \int_{l_0}^{l} (\widetilde{K}_{\chi m}\Psi^{i-1}_{\xi n} - \widetilde{K}_{\tau m}\Psi^{i-1}_{zn})\,dz, \\ \beta_{mp1} = c_m \int_{l_0}^{l} \widetilde{K}_{\chi m} B_{p1}\,dz, \qquad \beta_{mp2} = c_m \int_{l_0}^{l} \widetilde{K}_{\chi m} B_{p2}\,dz, \\ \beta_{m\omega1} = c_m \int_{l_0}^{l} \widetilde{K}_{\chi m} B_{\omega1}\,dz, \qquad \beta_{m\omega2} = c_m \int_{l_0}^{l} \widetilde{K}_{\chi m} B_{\omega2}\,dz, \\ \beta_{m\varphi1} = c_m \widetilde{\Phi}_{Am}, \qquad \beta_{m\varphi2} = c_m \widetilde{\Phi}_{Cm}, \end{gathered} \tag{53}$$

and

$$c^{-1}_m = \int_{l_0}^{l} (-\widetilde{\Phi}_{\xi m}\chi_{\xi m} + \Phi_{zm}\tau_m)\,dz - \widetilde{\Phi}_{Am}\varphi_{1m} - \widetilde{\Phi}_{Cm}\varphi_{2m}. \tag{54}$$

By introducing (52) into expressions for the moments (43) we obtain

$$\begin{gathered} M^i_{\xi n} = -\bar{\varphi}^i_{1n}(p^2_{ni}B_{p\xi1} - \omega^2 B_{\omega\xi1} - B_{\varphi\xi1}) - \bar{\varphi}^i_{2n}(p^2_{ni}B_{p\xi2} - \omega^2 B_{\omega\xi2} - B_{\varphi\xi2}) + \\ + p^2_{ni}\Phi^{*i-1}_{\xi n} - \omega^2\Psi^{*i-1}_{\xi n}, \\ M^i_{zn} = -\bar{\varphi}^i_{1n}(p^2_{ni}B_{pz1} - \omega^2 B_{\omega z1} - B_{\varphi z1}) - \bar{\varphi}^i_{2n}(p^2_{ni}B_{pz2} - \omega^2 B_{\omega z2} - B_{\varphi z2}) + \\ + p^2_{ni}\Phi^{*i-1}_{zn} - \omega^2\Psi^{*i-1}_{zn}, \end{gathered} \tag{55}$$

where

$$\begin{gathered} \Phi^*_{\xi n} = \Phi_{\xi n} + \sum_{m=1}^{n-1}\beta_{m\Phi}M_{\xi m}, \qquad \Psi^*_{\xi n} = \Psi_{\xi n} + \sum_{m=1}^{n-1}\beta_{m\Psi}M_{\xi m}, \\ \Phi^*_{zn} = \Phi_{zn} + \sum_{m=1}^{n-1}\beta_{m\Phi}M_{zm}, \qquad \Psi^*_{zn} = \Psi_{zn} + \sum_{m=1}^{n-1}\beta_{m\Psi}M_{zm}, \end{gathered} \tag{56}$$

and

$$\begin{aligned} B_{p\xi 1} &= B_{p1} + \sum_{m=1}^{n-1} \beta_{mp1} M_{\xi m}, & B_{pz1} &= \sum_{m=1}^{n-1} \beta_{mp1} M_{zm}, \\ B_{\omega\xi 1} &= B_{\omega 1} + \sum_{m=1}^{n-1} \beta_{m\omega 1} M_{\xi m}, & B_{\omega z1} &= \sum_{m=1}^{n-1} \beta_{m\omega 1} M_{zm}, \\ B_{\varphi\xi 1} &= \sum_{m=1}^{n-1} \beta_{m\varphi 1} M_{\xi m}, & B_{\varphi z1} &= \sum_{m=1}^{n-1} \beta_{m\varphi 1} M_{zm}. \end{aligned} \tag{57}$$

(Formulae from $B_{p\xi 2}$, $B_{\omega\xi 2}$, etc., are analogous).

Apparently both the coefficients β_{mp}, $\beta_{m\omega}$, $\beta_{m\varphi}$ and the corresponding functions $B_{p\xi}$, $B_{\omega\xi}$, $B_{\varphi\xi}$, B_{pz}, $B_{\omega z}$, $B_{\varphi z}$ are the same for all approximations of the n^{th} form. The moments $M^i_{\xi n}$, M^i_{zn} defined by equation (55) satisfy the orthogonality conditions for all of the preceding forms at any values of p^2_{ni}, $\bar{\varphi}^i_{1n}$ and $\bar{\varphi}^i_{2n}$. The quantity p^2_{ni} is found from the simplest condition of the passed integral equality of the moments of the $(i-1)^{th}$ and i^{th} approximations for which the work of the loads of the $(i-1)^{th}$ and of the i^{th} approximation for displacement of the $(i-1)^{th}$ approximation are equated, which gives /304

$$\begin{aligned} &\int_{l_0}^{l} (-\chi^{i-1}_{\xi n} M^i_{\xi n} + \tau^{i-1}_n M^i_{zn})\, dz - \varphi^{i-1}_{1n} M^i_{An} - \varphi^{i-1}_{2n} M^i_{Cn} = \\ &= \int_{l_0}^{l} (-\chi^{i-1}_{\xi n} M^{i-1}_{\xi n} + \tau^{i-1}_n M^{i-1}_{zn})\, dz - \varphi^{i-1}_{1n} M^{i-1}_{An} - \varphi^{i-1}_{2n} M^{i-1}_{Cn}. \end{aligned} \tag{58}$$

Using more complex conditions in this problem (for example, evaluation of the weighted least square fit (ref. 4)), complicates the solution practically without improving the convergence.

By substituting expression (55) for the moments into equation (58), we arrive at the following equation which correlates quantities p^2_{ni} with $\bar{\varphi}^i_{1n}$ and $\bar{\varphi}^i_{2n}$:

$$-p^2_{ni} = \frac{a^i_n - (b^i_{1n}\bar{\varphi}^i_{1n} + b^i_{2n}\bar{\varphi}^i_{2n})}{c^i_n - (d^i_{1n}\bar{\varphi}^i_{1n} + d^i_{2n}\bar{\varphi}^i_{2n})}, \tag{59}$$

where

$$
\begin{aligned}
a^i_n &= \int_{l_0}^{l} [-\chi^{i-1}_{\xi n}(M^{i-1}_{\xi n}+\omega^2\Psi^{*i-1}_{\xi n})+\tau^{i-1}_n(M^{i-1}_{zn}+\omega^2\Psi^{*i-1}_{zn})]\,dz- \\
&\quad -\varphi^{i-1}_{1n}(M^{i-1}_{An}+\omega^2\Psi^{*i-1}_{An})-\varphi^{i-1}_{2n}(M^{i-1}_{Cn}+\omega^2\Psi^{*i-1}_{Cn}), \\
c^i_n &= \int_{l_0}^{l} (-\chi^{i-1}_{\xi n}\Phi^{*i-1}_{\xi n}+\tau^{i-1}_n\Phi^{*i-1}_{zn})\,dz-\varphi^{i-1}_{1n}\Phi^{*i-1}_{An}-\varphi^{i-1}_{2n}\Phi^{*i-1}_{Cn}, \\
b^i_{1n} &= \int_{l_0}^{l} [-\chi^{i-1}_{\xi n}(B_{\varphi\xi 1}+\omega^2 B_{\omega\xi 1})+\tau^{i-1}_n(B_{\varphi z1}+\omega^2 B_{\omega z1})]\,dz- \\
&\quad -\varphi^{i-1}_{1n}(B_{\varphi\xi 1,A}+\omega^2 B_{\omega\xi 1,A})-\varphi^{i-1}_{2n}(B_{\varphi\xi 1,C}+\omega^2 B_{\omega\xi 1,C}), \\
b^i_{2n} &= \int_{l_0}^{l} [-\chi^{i-1}_{\xi n}(B_{\varphi\xi 2}+\omega^2 B_{\omega\xi 2})+\tau^{i-1}_n(B_{\varphi z2}+\omega^2 B_{\omega z2})]\,dz- \\
&\quad -\varphi^{i-1}_{1n}(B_{\varphi\xi 2,A}+\omega^2 B_{\omega\xi 2,A})-\varphi^{i-1}_{2n}(B_{\varphi\xi 2,C}+\omega^2 B_{\omega\xi 2,C}), \\
d^i_{1n} &= \int_{l_0}^{l} (-\chi^{i-1}_{\xi n}B_{p\xi 1}+\tau^{i-1}_n B_{pz1})\,dz-\varphi^{i-1}_{1n}B_{p\xi 1,A}-\varphi^{i-1}_{2n}B_{p\xi 1,C}, \\
d^i_{2n} &= \int_{l_0}^{l} (-\chi^{i-1}_{\xi n}B_{p\xi 2}+\tau^{i-1}_n B_{pz2})\,dz-\varphi^{i-1}_{1n}B_{p\xi 2,A}-\varphi^{i-1}_{2n}B_{p\xi 2,C}.
\end{aligned}
\tag{60}
$$

here Φ^*_{An}, Ψ^*_{An}, $B_{p\xi 1,A}$, $B_{p\xi 2,A}$, $B_{\omega\xi 1,A}$, $B_{\omega\xi 2,A}$, $B_{\varphi\xi 1,A}$, $B_{\varphi\xi 2,A}$ are values of the corresponding functions at point A and analogously Φ^*_{Cn} etc. are their values at point C. On the other hand, from the boundary conditions (21) and (29), taking into account (23), it follows that (when $\omega^2 \neq 0$)

$$
\begin{aligned}
&\bar\varphi^i_{1n}(e_1 q^2_{ni}-f_1)+\bar\varphi^i_{2n}(e_2 q^2_{ni}-f_2)=q^2_{ni}\Phi^{i-1}_{An}-\Psi^{i-1}_{An} \\
&\bar\varphi^i_{1n}(g_1 q^2_{ni}-h_1)+\bar\varphi^i_{2n}(g_2 q^2_{ni}-h_2)=(q^2_{ni}+1)\Psi^{i-1}_{An}, \\
&\qquad\left(q^2_{ni}=\frac{p^2_{ni}}{\omega^2}\right),
\end{aligned}
\tag{61}
$$

where /305

$$
\begin{aligned}
&e_1=\varrho_x J_A, && e_2=\varrho_x(J_A-2\varrho S_A), \\
&f_1=\varrho_x(r_A S_A+\varrho\gamma_1 P_{1x}), && f_2=\varrho_x r_C S_A, \\
&g_1=\varrho_x r_A S_A, && g_2=\varrho_x r_A S_C, \\
&h_1=\varrho_x r_A(0{,}5\gamma_1 P_{1x}-S_A), && h_2=\varrho_x r_A[(1-0{,}5\gamma_2)P_{1x}-S_C].
\end{aligned}
\tag{62}
$$

Having solved equation (61) with respect to angles $\bar{\varphi}^{i}_{1n}$ and $\bar{\varphi}^{i}_{2n}$ we find

$$\begin{gathered}\bar{\varphi}^{i}_{1n}=\frac{1}{\Delta_{ni}}\left\{q^{4}_{ni}(g_2\Phi^{i-1}_{An}-e_2\Psi^{i-1}_{An})-\right.\\ \left.-q^{2}_{ni}\left[h_2\Phi^{i-1}_{An}+(g_2+e_2-f_2)\Psi^{i-1}_{An}\right]+(h_2+f_2)\Psi^{i-1}_{An}\right\},\\ \bar{\varphi}^{i}_{2n}=\frac{-1}{\Delta_{ni}}\left\{q^{4}_{ni}(g_1\Phi^{i-1}_{An}-e_1\Psi^{i-1}_{An})-\right.\\ \left.-q^{2}_{ni}\left[h_1\Phi^{i-1}_{An}+(g_1+e_1-f_1)\Psi^{i-1}_{An}\right]+(h_1+f_1)\Psi^{i-1}_{An}\right\},\end{gathered} \tag{63}$$

where

$$\begin{gathered}\Delta_{ni}=q^{4}_{ni}(e_1g_2-g_1e_2)-q^{2}_{ni}(e_1h_2+f_1g_2-g_1f_2-h_1e_2)+\\ +(f_1h_2-h_1f_2).\end{gathered} \tag{64}$$

By substitution of (63) into (59) we arrive at a cubical equation with respect to p^{2}_{ni}. Having found by the ordinary method its real loads (see for example (ref. 7)) and having determined the corresponding values of angles, we take as the solution for further calculation that group of values of p^{2}_{ni}, $\bar{\varphi}^{i}_{1n}$, $\bar{\varphi}^{i}_{2n}$ for which the quantity $(\varphi^{i}_{1n}-\varphi^{i-1}_{1n})^2+(\varphi^{i}_{2n}-\varphi^{i-1}_{2n})^2$ becomes the smallest.

In calculations it is useful to keep in mind that

$$e_1g_2-g_1e_2=2\varrho r_A(S^{2}_{A}-J_AV)=-2\varrho r_AVJ_{\tau}. \tag{65}$$

From the found values of p^{2}_{ni}, $\bar{\varphi}^{i}_{ni}$ and $\bar{\varphi}^{i}_{2n}$ we determine moments $M^{i}_{\xi n}$ and M^{i}_{zn} using formulae (55) and angles φ^{i}_{1n} and φ^{i}_{2n} from formulae (44) and (52).

It is well known (ref. 4) that the above presented simple iteration problem converges only when the flexibility parameter of the blade $\nu<1$. The flexibility parameter

$$\nu=\frac{\omega^2}{\omega^2_{*}}, \tag{66}$$

where ω^* is the least nonzero eigenvalue of the corresponding stability problem, the equations for which are obtained from the equations of the previous problem if one assumes in them that $p^2 = 0$ and $\omega^2 = -\omega_*^2$, from which[1]

$$\begin{aligned} M_\xi &= \omega_*^2 \widetilde{\Psi}_\xi, \\ M_z &= \omega_*^2 \Psi_z, \end{aligned} \tag{67}$$

where

$$\widetilde{\Psi}_\xi = \Psi_\xi - (\varphi_1 B_{\omega 1} + \varphi_2 B_{\omega 2}),$$

under boundary conditions ($\omega_* \neq 0$, $\varphi_1 \neq 0$, $\varphi_2 \neq 0$) /306

$$\left.\begin{aligned} f_1\varphi_1 + f_2\varphi_2 &= \Psi_A, \\ h_1\varphi_1 + h_2\varphi_2 &= -\Psi_A, \end{aligned}\right\} \tag{68}$$

from which

$$\widetilde{\Psi}_\xi = \Psi_\xi - \frac{(f_2 + h_2) B_{\omega 1} - (f_1 + h_1) B_{\omega 2}}{f_1 h_2 - f_2 h_1} \Psi_A. \tag{69}$$

The system of equations (67) is solved by means of a simple iteration

$$\begin{aligned} M_\xi^l &= \omega_{*l}^2 \widetilde{\Psi}_\xi^{l-1}, \\ M_z^l &= \omega_{*l}^2 \Psi_z^{l-1}, \end{aligned} \tag{70}$$

where

$$\omega_{*l}^2 = \frac{\int_{l_0}^{l} \left(-\chi_\xi^{l-1} M_\xi^{l-1} + \tau^{l-1} M_z^{l-1}\right) dz}{\int_{l_0}^{l} \left(-\chi_\xi^{l-1} \widetilde{\Psi}_\xi^{l-1} + \tau^{l-1} \Psi_z^{l-1}\right) dz}. \tag{71}$$

---

[1] Apparently the given formulation of the stability problem has only a mathematical significance since in the case of real changes of the direction of longitudinal forces the contact conditions in the hinge point would change.

Let us note that when $\varphi_2 = 0$

$$\tilde{\Psi}_\xi = \Psi_\xi - \frac{B_{\omega 1}}{B_{\omega 1A} + B_{Q1}} \Psi_A, \tag{72}$$

and when $\varphi_1 = 0$

$$\tilde{\Psi}_\xi = \Psi_\xi - \frac{B_{\omega 2}}{B_{\omega 2C} + B_{Q2}} \Psi_C. \tag{72'}$$

For the calculation of ω_*^2, 2-3 approximations are generally sufficient.

Knowing the value of ω_*^2 we find ν, and if $\nu \geq 1$, for the i^{th} approximation we take

$$\begin{aligned} M_{\xi n}^i &= \tilde{M}_{\xi n}^i - \varepsilon\left(\tilde{M}_{\xi n}^i - M_{\xi n}^{i-1}\right), \qquad & \varphi_{1n}^i &= \tilde{\varphi}_{1n}^i - \varepsilon\left(\tilde{\varphi}_{1n}^i - \varphi_{1n}^{i-1}\right), \\ M_{zn}^i &= \tilde{M}_{zn}^i - \varepsilon\left(\tilde{M}_{zn}^i - M_{zn}^{i-1}\right), \qquad & \varphi_{2n}^i &= \tilde{\varphi}_{2n}^i - \varepsilon\left(\tilde{\varphi}_{2n}^i - \varphi_{2n}^{i-1}\right), \end{aligned} \tag{73}$$

where $\tilde{M}_{\xi n}^i$, $\tilde{M}_{zn}^i$, $\tilde{\varphi}_{1n}^i$, $\tilde{\varphi}_{2n}^i$ are the quantities determined above by means of a simple iteration.

ε is the coefficient which insures convergence when $\nu \geq 1$, which according to (ref. 4) can be taken as equal to

$$\varepsilon = \frac{\nu}{1+\nu}. \tag{74}$$

In order to improve the convergence and the uniformity of the process of solution it is worthwhile to make use of formulae (73) and (74) also when $\nu < 1$. When selecting the initial values for these quantities the following must be kept in mind: since the first form of vibrations generally is close to the first harmonic oscillation we may take $M_{\xi 1}^0 = M_{MI}$, i.e.

$$M_{\xi 1}^0 = -\varphi_1^0\left[p_{M1}^2 B_{p1} - \omega^2 B_{\omega 1} + \frac{\varphi_{2MI}}{\varphi_{1MI}}\left(p_{M1}^2 B_{p2} - \omega^2 B_{\omega 2}\right)\right]. \tag{75}$$

where p_{MI} and $\varphi_{2MI}/\varphi_{1MI}$ are the frequency and the ratio of angles for the first harmonic form, determined by solving system (21) when $\Phi_L = \Psi_L = 0$. The quantity φ_1^0 is random. It may be taken to be such that $M_{A1}^0 = 1$. In calculation of the second form we assume that /307

$$\begin{aligned} M_{\xi 2}^0 &= M_\xi^0 + \beta_1^0 M_{\xi 1}, \quad & \varphi_{12}^0 &= \varphi_1^0 + \beta_1^0 \varphi_{11}, \\ M_{z2}^0 &= M_z^0 + \beta_1^0 M_{z1}, \quad & \varphi_{22}^0 &= \varphi_2^0 + \beta_1^0 \varphi_{21}, \end{aligned} \tag{76}$$

where M_ξ^0, M_z^0 are some functions;

φ_1^0, φ_2^0 are the corresponding angles as will be shown below;

$M_{\xi 1}$, M_{z1}, φ_{11}, φ_{21} are moments and angles during oscillations in accordance with the first form and are considered to be known.

By orthogonalizing (76) to the first form we find

$$\beta_1^0 = -c_1 \left[\int_{l_0}^{l} \left(-\chi_\xi^0 \widetilde{\Phi}_{\xi 1} + \tau^0 \Phi_{z1} \right) - \varphi_1^0 \widetilde{\Phi}_{A1} - \varphi_2^0 \widetilde{\Phi}_{C1} \right], \tag{77}$$

where χ_ξ^0, τ^0 are expressed through M_ξ^0, M_z^0 using formulae (4).

Since in the general case the predominant deformation in the second form of vibrations can be both the bending and torsional, it is worthwhile to evaluate in the first approximation 2 values of p_{21}^2, assuming:

a) predominantly torsional deformations

$$\begin{aligned} &M_\xi^0 = 0, \quad \varphi_1^0 = \varphi_2^0 = 0, \\ &M_z^0 = 1 - \left(\frac{z - l_0}{l - l_0} \right)^2, \end{aligned} \tag{78}$$

which satisfies conditions $M_z^0(l) = 0$ and $\theta(l_0) = 0$;

b) predominantly bending deformation

$$\begin{aligned} &M_\xi^0 = \frac{M_{C1} z - M_{A1} \tilde{z} (1 - 2\varrho/z^*)(1 - 2\varrho/l)^2}{2\varrho (1 - 2\varrho/z^*)(1 - 2\varrho/l)^2} (1 - z/l)^2 (1 - z/z^*), \\ &M_z^0 = 0. \end{aligned} \tag{79}$$

The expression for the moment M_ξ^0 satisfies boundary conditions $M_\xi^0(l) = M_\xi^{0\prime}(l) = 0$. It changes sign when $z = z^*$ and insures that the following equalities are fulfilled: $M_A^0 = M_{A1}$, $M_C^0 = M_{C1}$. From formulae (27) it is apparent that here $\varphi_1^0 = \varphi_{11}$, $\varphi_2^0 = \varphi_{21}$. Assuming $\rho \ll l$ and expressing M_{C1} according to (30) as

$$M_{C1} = M_{A1} - 2\rho\,[P_{z\pi}(\varphi_{11} + \varphi_{21}) - Q_{A1}],$$

we obtain

$$M_\xi^0 = \frac{M_{A1}(1 + z/z^*) - [P_{z\pi}(\varphi_{11} + \varphi_{21}) - Q_{A1}]\,z}{1 - 2\rho/z^*}(1 - z/l)^2(1 - z/z^*). \tag{80}$$

After the determination of the eigenvalue of p_{21}^2 for each variation further calculations of the second form are conducted on the basis of initial variation in which the quantity p_{21}^2 had the smallest value.

In calculation of the third form we take

$$M_{\xi 3}^0 = M_\xi^0 + \beta_1^0 M_{\xi 1} + \beta_2^0 M_{\xi 2} \tag{81}$$

etc. where for the values of M_ξ^0, M_z^0 and φ_1^0, φ_2^0 we take the same quantities as for the second form but a correspondingly larger value of p_{21}^2. The coefficients β_1^0, β_2^0 were determined from formula (77) -- in the latter case with a change of quantities C_1, $\tilde{\Phi}_{\xi 1}$, etc. for C_2, $\tilde{\Phi}_{\xi 2}$ etc., respectively.

Analogously one can assign initial approximations for subsequent forms, /308 but as a result of assumptions which were taken as the basis of the method, calculations of higher forms are of limited practical significance.

If by analogy with the previous calculations or on the basis of other considerations the predominant nature of deformation of the second and of the third form are known then the initial approximation is taken directly for the appropriate variation. In calculation of ω_*^2 for initial values one may also take $M_\xi^0 = M_{MI}$, $M_z^0 = 0$. For the stationary blade $M_A = M_C = 0$, and the initial approximation for M_ξ should be assigned in the form

$$M_\xi^0 = \left(\frac{z}{l}\right)^s \left(1-\frac{z}{l}\right)^2 \left(1-\frac{z}{z^*}\right)^t, \tag{82}$$

where the power s is equal to the number of degrees of freedom in the hinge [when s = 2, from (82) it follows that $Q_A = 0$, which corresponds to condition (30). In other words, when $\omega = 0$ the pin blade with 2 degrees of freedom behaves as a beam with free ends]. For the second (first zero) form t = 0 and for the third form t = 1.

The proposed general calculation method was programmed for the electronic digital computer M20. Let us consider certain numerical calculation methods when certain simplifications are possible.

1. Blade with one degree of freedom in the pinned joint (for the specific case we assume that $\varphi_2 = 0$, $\varphi = \varphi_1$). From (14) it follows that[1]

$$\begin{aligned} M_\xi &= -\varphi(p^2 B_p - \omega^2 B_\omega) + p^2 \Phi_\xi - \omega^2 \Psi_\xi, \\ M_z &= p^2 \Phi_z - \omega^2 \Psi_z, \end{aligned} \tag{83}$$

under boundary condition (34) and orthogonality condition

$$\int_{l_0}^{l} (-\widetilde{\Phi}_{\xi m} \gamma_{\xi n} + \Phi_{zm} \tau_n)\, dz - \widetilde{\Phi}_{Am} \varphi_n = 0. \tag{84}$$

Since when $\varphi_2 = 0$ for calculation of p_{ni}^2 formula (59) becomes

$$p_{ni}^2 = \frac{a_n^i - b_n^i \bar{\varphi}_n^i}{c_n^i - d_n^i \bar{\varphi}_n^i}, \tag{85}$$

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[1]Subscript 1 of B_p, B_ω and other quantities was omitted here.

by introducing (85) into expression (34), written for the determination of $\overline{\varphi}_n^i$, we shall obtain a quadratic equation with respect to $\overline{\varphi}_n^i$:

$$A_n^i(\overline{\varphi}_n^i)^2 - B_n^i\overline{\varphi}_n^i + C_n^i = 0, \tag{86}$$

where

$$\begin{aligned} A_n^i &= B_{pA}b_n^i - \omega^2(B_{\omega A} + B_Q)d_n^i, \\ B_n^i &= B_{pA}a_n^i - \omega^2(B_{\omega A} + B_Q)c_n^i + \Phi_{An}^{i-1}b_n^i - \omega^2\Psi_{An}^{i-1}d_n^i, \\ C_n^i &= \Phi_{An}^{i-1}a_n^i - \omega^2\Psi_{An}^{i-1}c_n^i. \end{aligned} \tag{87}$$

The sought value $\overline{\varphi}_n^i$ gives that root of the equation (86) for which the quantity $(\varphi_n^i - \varphi_n^{i-1})^2$ is minimal.

2. Instead of determining the quantity p_{ni}^2 from the conditions of integral closeness of the i^{th} to the $(i-1)^{th}$ approximations, generally speaking it is possible to make use of the method in which maximum values of moments are compared by assuming that $M_{\xi n}^i = M_{\xi n}^{i-1}$ at the point where $M_{\xi n}^{i-1}$ has its maximum value. However, the convergence of such a process is not always satisfactory, and in the general case this method does not lead to useful simplifications. However, in a specific case for the blade with one degree of freedom in a pinned joint, when $M_{max} = M_A \neq 0$, i.e. when $\omega^2 B_\rho \neq 0$, and when according to formula (27) /309

$$\varphi = -\frac{M_A}{\omega^2 B_Q}, \tag{88}$$

by equating $M_{An}^i = M_{An}^{i-1} = M_{An}^0$ (or $\overline{\varphi}_n^i = \varphi_n^{i-1} = \varphi_n^0$) we obtain

$$-\frac{M_{An}^0}{\omega^2 B_Q} = \frac{p_{ni}^2\Phi_{An}^{i-1} - \omega^2\Psi_{An}^{i-1}}{p_{ni}^2 B_{pA} - \omega^2(B_{\omega A} + B_Q)},$$

from which

$$q_{ni}^2 = \frac{p_{ni}^2}{\omega^2} = \frac{\frac{M_{An}^0}{\omega^2 B_Q}(B_{\omega A} + B_Q) + \Psi_{An}^{i-1}}{\frac{M_{An}^0}{\omega^2 B_Q}B_{pA} + \Phi_{An}^{i-1}}. \tag{89}$$

By selecting the scale of moments to be such that $M_{An}^{0} = \omega^2 B_{\varrho}$ (i.e. $\varphi_n^0 = -1$), we obtain a very simple formula

$$q_{ni}^2 = \frac{B_{\omega A} + B_{\varrho} + \Psi_{An}^{i-1}}{B_{pA} + \Phi_{An}^{i-1}} = \frac{B_{\varrho} + \tilde{\Psi}_{An}^{i-1}}{\tilde{\Phi}_{An}^{i-1}}. \qquad (90)$$

By introducing the found value p_{ni}^2 and $\bar{\varphi}_n^i = -1$ into equation (83) we can determine the values of moments which must be further integrated to the preceeding forms (just as it is done for moments of the initial approximation) and normalized for the condition $\varphi_n^i = -1$.

For the first frequency, when $\Psi_{A1} \ll B_{\omega A}$, $\Phi_{A1} \ll B_{pA}$, calculations by formula (90) generally converge rapidly. In calculating the subsequent forms, despite the simplicity of the method, its practical application is difficult due to loss of accuracy in calculation of the denominator of the formula (90) which is a small difference of large numbers.

The accuracy of calculation can be somewhat increased by determining $\tilde{\Phi}_A$ when $\varphi = -1$ directly from the expression

$$\tilde{\Phi}_A = \varrho_n \int_0^l \int_z^l F \int_0^z \left(1 - \int_0^z \gamma_\xi \cos\alpha \, dz\right) dz^3. \qquad (91)$$

3. For a nonrotating blade, when $\omega = 0$ we can find from the system (61) the following

$$\begin{aligned} \bar{\varphi}_{1n}^i &= \frac{g_2 \Phi_{An}^{i-1} - e_2 \Psi_{An}^{i-1}}{e_1 g_2 - g_1 e_2}, \\ \bar{\varphi}_{2n}^i &= -\frac{g_1 \Phi_{An}^{i-1} - e_1 \Psi_{An}^{i-1}}{e_1 g_2 - g_1 e_2}. \end{aligned} \qquad (92)$$

Substituting values for $\bar{\varphi}_{1n}^i$ and $\bar{\varphi}_{2n}^i$ into expressions (55) we obtain /310

$$\begin{aligned} M_{\xi n}^i &= p_{ni}^2 \tilde{\Phi}_{\xi n}^{*i-1}, \\ M_{zn}^i &= p_{ni}^2 \tilde{\Phi}_{zn}^{*i-1}, \end{aligned} \qquad (93)$$

where

$$\widetilde{\Phi}^{*i-1}_{\xi n} = \Phi^{*i-1}_{\xi n} - (\overline{\varphi}^{i}_{1n} B_{p\xi 1} + \overline{\varphi}^{i}_{2n} B_{p\xi 2}),$$
$$\widetilde{\Phi}^{*i-1}_{zn} = \Phi^{*i-1}_{zn} - (\overline{\varphi}^{i}_{1n} B_{pz1} + \overline{\varphi}^{i}_{2n} B_{pz2}). \qquad (94)$$

Here when $\omega = 0$ it follows from (27) that $M_A = M_C = 0$ and, consequently, $\widetilde{\Phi}_A = \widetilde{\Phi}_C = 0$, $\beta_{\varphi 1} = \beta_{\varphi 2} = 0$, $B_{\varphi\xi} = B_{\varphi z} = 0$.

Formula (59) takes on the following form (since $b^i_{1n} = b^i_{2n} = 0$)

$$p^2_{ni} = \frac{a^i_n}{c^i_n} = \frac{\int\limits_{l_0}^{l} \left(-\chi^{i-1}_{\xi n} M^{i-1}_{\xi n} + \tau^{i-1}_n M^{i-1}_{zn}\right) dz}{\int\limits_{l_0}^{l} \left(-\chi^{i-1}_{\xi n} \widetilde{\Phi}^{*i-1}_{\xi n} + \tau^{i-1}_n \widetilde{\Phi}^{*i-1}_{zn}\right) dz}. \qquad (95)$$

4. For an idealized blade with rigid point hinge at point r_A ($\rho = 0$, $\varphi_2 = 0$, $\varphi_1 = \varphi$) two boundary conditions (21) become a single condition $M_A = 0$, from which

$$\varphi = \frac{p^2 \Phi_A - \omega^2 \Psi_A}{p^2 B_{pA} - \omega^2 B_{\omega A}}. \qquad (96)$$

The first equation (14) becomes

$$M_\xi = p^2 \Phi_\xi - \omega^2 \Psi_\xi - \frac{p^2 \Phi_A - \omega^2 \Psi_A}{p^2 B_{pA} - \omega^2 B_{\omega A}} (p^2 B_p - \omega^2 B_\omega). \qquad (97)$$

The last equation applicable to weakly twisted blades and its solution by the method of comparing maximum ordinates was considered by I. A. Birger (ref. 4).

5. In calculation of the first form of vibrations normally $M_z \ll M_\xi$, so that

$$\chi_{\xi 1} \approx -\delta_{11} M_{\xi 1},$$
$$\tau_1 \approx \delta_{12} M_{\xi 1} = -\frac{\delta_{12}}{\delta_{11}} \chi_{\xi 1}, \qquad (98)$$

i.e., in the determination of the first frequency and corresponding linear displacements u and v it is possible to neglect the effect of torsional moments, however the bending deformation is accompanied, just as for rigidly mounted blades (ref. 3), by turning of cross sections,

$$\theta_1 \approx -\int_0^z \frac{\delta_{12}}{\delta_{11}} \chi_{\xi 1}\, dz. \tag{99}$$

Let us also note that the term Ψ_z normally has secondary significance because of relatively weak effects of centrifugal forces on predominantly twisting /311 forms of vibrations.

6. Assuming in all equations that $\varphi_1 = \varphi_2 = 0$ we shall arrive at a system of equations

$$\begin{aligned} M_{\xi n}^i &= p_{ni}^2 \Phi_{\xi n}^{*i-1} - \omega^2 \Psi_{\xi n}^{*i-1}, \\ M_{zn}^i &= p_{ni}^2 \Phi_{zn}^{*i-1} - \omega^2 \Psi_{zn}^{*i-1} \end{aligned} \tag{100}$$

when

$$p_{ni}^2 = \frac{a_n^i}{c_n^i}, \tag{101}$$

which describes vibrations of a twisted blade rigidly mounted along the base cross-section, considered earlier in (refs. 2,3,5).

7. In calculation of the first form of vibrations of the blade with two degrees of freedom in a pinned joint instead of a direct solution of a cubical equation one can successfully solve the systems of equations (59) and (61) by the method of successive approximations.

For the solution of this system, in the beginning one should assume that $\varphi_{1n}^i = \bar{\varphi}_{1n}^{i-1}$, $\bar{\varphi}_{2n}^i = \varphi_{2n}^{i-1}$, determine quantities p_{ni}^2 and q_{ni}^2 from (59) and then from this value of q_{ni}^2 find more exact values of $\bar{\varphi}_{1n}^i$ and $\bar{\varphi}_{2n}^i$ from formulae (63) and (64).

Then p_{ni}^2 is recalculated from (59), half-sum of the initial and the more exact values of p_{ni}^2, from which again angles are determined, etc. This process converges rapidly, as a rule, but calculations must be carried out with sufficient accuracy.

As an example, several frequencies were calculated for the pinned blade described in the work of S. M. Grinberg (ref. 6), which appears in this collection, where the variation method for the solution of this problem is presented.

In calculating vibrations of the blade by the method of successive approximations the starting data must have:

a) geometrical characteristics: radii r_A, r_C, r_0, length of the fin l_Π, cross sectional parameters of the fin $F(z)$, $J_p(z)$, the angle of mounting of the cross sections $\alpha(z)$, characteristics of the tail section V_{HA}, S_{HA}, J_{HA}, and dimensions of the pinned joint ρ, ρ_1, ρ_2;

b) density of the material of the blade ρ_Π;

c) yield coefficients $\delta_{11}(z)$, $\delta_{22}(z)$, $\delta_{12}(z)$;

d) angular velocity of the rotor ω.

It is convenient to conduct calculations in the following sequence: first the second and the third forms of vibrations are determined when $\omega = 0$, then in sequence pendulum forms are calculated, value ω_*, first, second and third forms when $\omega \neq 0$.

Figures 5-8 show the convergence process of solutions for different calculation cases. In calculation of frequencies of a stationary blade for initial function purposely known rough approximations were taken: in calculation of the second form (fig. 5) $M^0_{\xi 2} = (z - l_0)(1 - \frac{z - l_0}{l_\Pi})^2$, which yields $M^0_{\xi 2}(l_0) = 0$ instead of the correct condition $M_A = 0$. In calculation of the third form (fig. 6) $M^0_{z3} = 1 - \frac{z - l_0}{l_\Pi}$. Despite this even in the first approximation the estimate of the frequency is sufficiently accurate. Calculation of the first form when $\omega = 10^3$ sec^{-1} was carried out by two methods--general (fig. 7) and the method using condition $M^i_A = M^{i-1}_A$. In the first case the accurate frequency value is obtained, naturally after a smaller number of approximations but the extent of

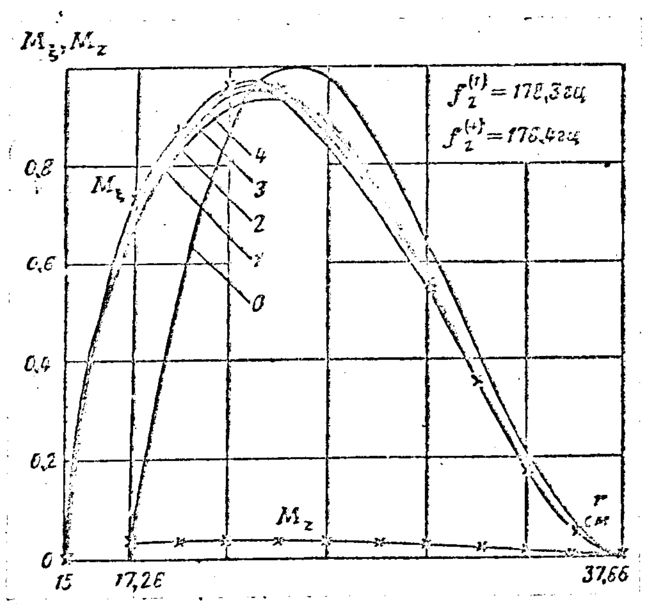

Figure 5. Projections of moments in calculation of the second form of vibrations of the stationary pinned blade, ω = 0; (0-4) number of approximation; -x- calculated on electronic digital computer.

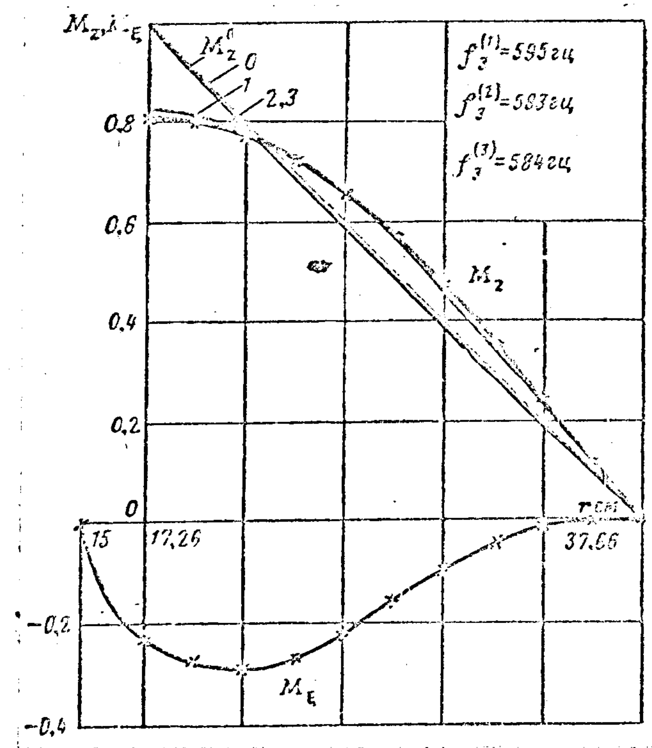

Figure 6. Projections of moments in calculation of the third form of vibrations of the stationary pinned blade, ω = 0; (0-3) number of approximation; -x- calculated on an electronic digital computer. /312

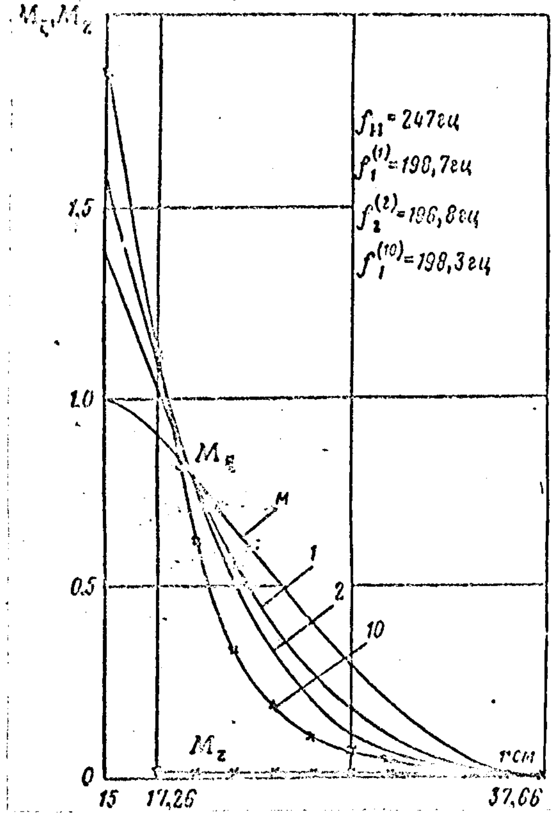

Figure 7. Projections of moments in calculation of the first form of vibrations of a rotating pinned blade. $\omega = 0^3$ sec^{-1}; M is "pendulum" form; (1-10) number of approximation; -x- calculated on an electronic digital computer.

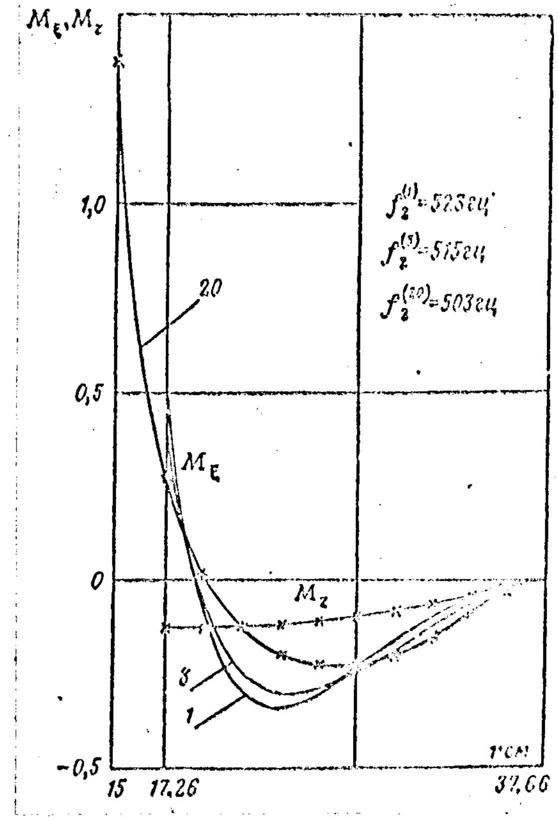

Figure 8. Projection of moments in calculation of the second form of vibrations of a rotating pinned blade, $\omega = 10^3$ sec^{-1}; (1-20) number of approximation; -x- calculated on an electronic digital computer.

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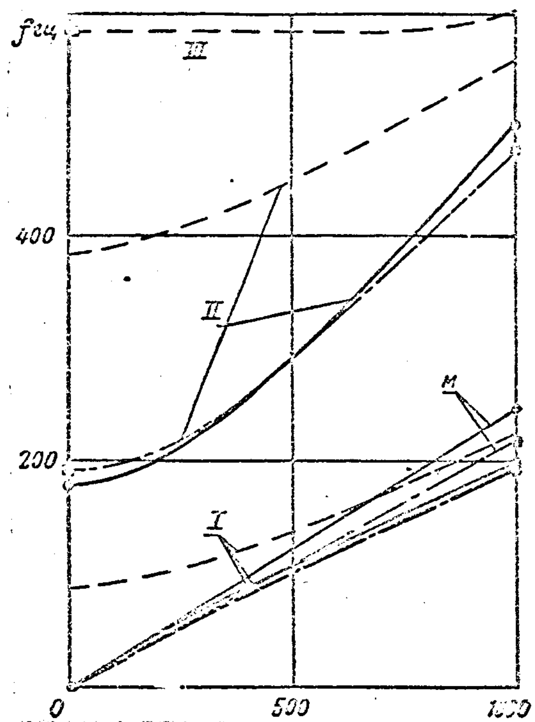

Figure 9. A comparison of frequency spectra of a pinned blade with one O-O and two - • - degrees of freedom in the hinge and of a corresponding rigidly mounted blade — — —; I, II, III are numbers of the forms of vibrations; M is the "pendulum" form.

each approximation increases. For the given blade the critical velocity $\omega_* = 0.184 \cdot 10^6$ sec^{-1}, so that when $\omega = 10^3$ sec^{-1} the flexibility parameter $\nu = 5.45$ and coefficient $\epsilon = 0.845$. In view of such a large flexibility of the blade, to obtain a satisfactory convergence along the projection of moments several approximations are required. The convergence of the solution of the second form of vibrations is shown in Figure 8.

The order of design of the blade with two degrees of freedom in the pinned joint is in general the same as shown above, with the exception that p_{ni}^2 and $\bar{\varphi}_{1n}^i$, $\bar{\varphi}_{2n}^i$ must be determined by somewhat different methods.

A comparison of the frequency spectra of a pinned blade with one and with two degrees of freedom in the pinned joint and a corresponding rigidly mounted blade (calculated on an electronic digital computer M20 according to reference 2) is shown in figure 9. The projections of dynamic bending, curvature and other characteristics of the pinned blades are shown in reference 6, where comparisons are made of the results calculated by the iteration and variation methods.

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